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Copositive Polynomial and Spline Approximation

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We prove that if a function $f \in \mathbf{C}[0, 1]$ changes sign finitely many times, then for any n large enough the degree of copositive approximation to f by quadratic spliners with $n-1$ equally spaced knots can be estimated by $C\omega_2(f, 1/n)$, where C is an absolute constant. We also show that the degree of copositive polynomial approximation to $f \in \mathbf{C}^1[0, 1]$ can be estimated by $Cn^{-1}\omega_r(f', 1/n)$, where the constant C depends only on the number and position of the points of sign change. This improves the results of Leviatan (1983, *Proc. Amer. Math. Soc.* **88**, 101–105) and Yu (1989, *Chinese Ann. Math.* **10**, 409–415), who assumed that for some $r \geqslant 1$, $f \in \mathbf{C}^r[0, 1]$. In addition, the estimates involved $Cn^{-r}\omega(f^{(r)}, 1/n)$ and the constant C dependended on the behavior of f in the neighborhood of those points. One application of the results is a new proof to our previous ω_2 estimate of the degree of copositive polynomia approximation of $f \in \mathbf{C}[0, 1]$, and another shows that the degree of copositive spline approximation cannot reach ω_4, just as in the case of polynomials.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbf{C}^r[0, 1]$ be the space of r times continuously differentiable functions on $[0, 1]$, and let $\mathbf{C}[0, 1]$ be the space of continuous functions. Let

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$f \in \mathbf{C}[0, 1]$; then denote by $\|\cdot\|$ the uniform norm taken over $[0, 1]$ and by $\omega_r(f, t)$ the usual rth modulus of smoothness of f.

We say that $f(x)$ changes sign at $y \in (0, 1)$ if $f(y) = 0$ and if for some small $\varepsilon > 0$, $\eta f(x) \geqslant 0$, if $y - \varepsilon < x < y$, and $\eta f(x) \leqslant 0$, if $y < x < y + \varepsilon$, where $\eta = \pm 1$. Such a y is called a point of sign change of f. We will assume that f has finitely many, say k, sign changes in $[0, 1]$. However, one should note that we do not exclude the possibility that f vanishes in some subinterval in which we may or may not designate (weak) sign changes. If f vanishes in $[a, b]$ and $y \in (a, b)$ is a point of sign change, then for sufficiently small ε, $[y - \varepsilon, y + \varepsilon] \subset [a, b]$ and contains no other point of sign change. Then we will write $\operatorname{sgn}(f(x)) := \eta \operatorname{sgn}(x - y)$ for $x \in [y - \varepsilon, y + \varepsilon]$. Our purpose is to estimate the degree of approximation of functions $f \in \mathbf{C}[0, 1]$, by means of polynomials p_n and splines s_n, which are copositive with f, i.e., polynomials p_n and splines s_n, which are nonnegative where f is and nonpositive where f is. If f does not vanish in any subinterval, then this is obviously characterized by $f(x)\, p_n(x) \geqslant 0$ and $f(x)\, s_n(x) \geqslant 0$ throughout $[0, 1]$. While the degree of nonnegative spline approximation to a nonnegative function is of the same order as the best spline approximation, to the best of our knowledge, nothing is known about the degree of copositive spline approximation to a function that changes its sign even once. Our first result gives an estimate on copositive spline approximation.

THEOREM 1. *Let* $f \in \mathbf{C}[0, 1]$ *change its sign* k *times at* $0 < y_1 < y_2 < \cdots < y_k < 1$, *and let* $\delta := \min_{1 \leqslant j \leqslant k-1}(y_{j+1} - y_j)$ *if* $k \geqslant 2$ *and* $\delta := 4$ *if* $k = 1$. *Then for each* $n \geqslant 4\delta^{-1}$, *there exists a quadratic spline* s_n *with* $n - 1$ *equally spaced interior knots which is copositive with* f, *such that*

$$\|f - s_n\| \leqslant C\omega_2(f, 1/n), \tag{1.1}$$

where C *is an absolute constant.*

Our next result is concerned with copositive polynomial approximation to a function $f \in \mathbf{C}^1[0, 1]$. Leviatan [lev] (for $r \leqslant 2$) and Yu [yu] (for $r > 2$) have shown that for any positive integer r, if $f \in \mathbf{C}^r[0, 1]$, and n is sufficiently large, then there exists a polynomial P_n copositive with f such that $\|f - P_n\| \leqslant Cn^{-r}\omega(f^{(r)}, 1/n)$, where C depends on the set of all points of sign change of f. But how large n should be depends on the behavior of f and its derivatives near those points of sign change. This dependency on f is unsatisfactory, and it turns out to be unnecessary. To see this we have the next result, which is stronger in that it merely assumes $f \in \mathbf{C}^1$ and the estimates involve the moduli of smoothness of f', with explicit description of the constants and the size of n.

640/80/2-6

THEOREM 2. *Let $f \in \mathbf{C}^1[0, 1]$, change its sign k times at $0 < y_1 < y_2 < \cdots < y_k < 1$, and let $\delta := \min_{0 \leqslant j \leqslant k}(y_{j+1} - y_j)$, where $y_0 := 0$ and $y_{k+1} := 1$. Then for any $r \geqslant 1$, there are positive constants $C_1 = C_1(k, r)$ and $C_2 = C_2(k, r, \delta)$, such that for each $n > C_1 \delta^{-1}$, there exists a polynomial P_n of degree $\leqslant n$ which is copositive with f and satisfies*

$$\|f - P_n\| \leqslant C_2 n^{-1} \omega_r(f', 1/n). \tag{1.2}$$

Remark. Note that while in Theorem 1, δ depends only upon the distance between the points of sign change, in Theorem 2, it depends also on their distance from the endpoints of the interval.

The more precise description of the size n enables us to apply results back and forth between copositive approximation by splines and that by polynomials. This is illustrated in the proof of the Theorem A and in an application of Theorm B cited below. We start with a new proof of a result from our previous paper [hly], namely,

THEOREM A. *Let $f \in \mathbf{C}[0, 1]$ change its sign k times at $0 < y_1 < y_2 < \cdots < y_k < 1$. Let $\delta := \min_{0 \leqslant j \leqslant k}(y_{j+1} - y_j)$, where $y_0 := 0$* and *$y_{k+1} := 1$. Then there are positive constants $C_1 = C_1(k)$ and $C_2 = C_2(k, \delta)$ such that for each $n > C_1 \delta^{-1}$, there exists a polynomial P_n of degree $\leqslant n$ which is compositive with f and satisfies*

$$\|f - P_n\| \leqslant C_2 \omega_2(f, 1/n).$$

Proof. Take the spline s_n obtained in Theorem 1. Then (1.1) yields

$$\omega_2(s_n, 1/n) \leqslant C\omega_2(f, 1/n).$$

But by Yu and Zhou [yz], if $s_n \in \mathscr{S}(m+1, n)$, where for a nonnegative integer m, $\mathscr{S}(m+1, n)$ denotes the collection of splines of order $m+1$ (i.e., piecewise polynomials of degree m in $\mathbf{C}^{m-1}[0, 1]$) on the partition $\mathbf{T} := \{i/n\}_{i=0}^{n}$, then

$$\omega_m(s_n', 1/n) \leqslant C(m)\, n\omega_{m+1}(s_n, 1/n). \tag{1.3}$$

Hence

$$\omega(s_n', 1/n) \leqslant Cn\omega_2(f, 1/n),$$

and we can apply Theorem 2 with $r = 1$ to s_n to obtain the desired polynomial. One should emphasize that we have to fix the value of n – the number of knots of s_n *and* the degree of P_n, simultaneously; of all of this before applying Theorem 1. This prohibits the dependency of n on the behavior of s_n in the second step. ∎

The following negative result by Songping Zhou [zhou] demonstrates that it is impossible to approximate a general function f by copositive polynomials at a rate of ω_4 even if this f has a continuous derivative and changes its sign only once in $[0, 1]$.

THEOREM B. *There is a function $f \in \mathbf{C}^1[0, 1]$ which changes its sign once in $[0, 1]$ and such that*

$$\limsup_{n \to \infty} \frac{E_n^{(0)}(f)}{\omega_4(f, 1/n)} = +\infty,$$

where $E_n^{(0)}(f)$ is the error of the best copositive approximation to f by polynomials of degree $\leqslant n$.

Applying Theorems 2 and B one can readily prove by contradiction that it is impossible to achieve the degree ω_4 for copositive approximation to a general function $f \in \mathbf{C}[0, 1]$ by splines $s \in \mathbf{C}^1[0, 1]$, with $n-1$ interior knots (not necessarily equally spaced) such that for some positive r

$$\omega_r(s', 1/n) \leqslant Cn\omega_4(f, 1/n).$$

For splines with equally spaced knots, we can be more explicit, namely,

THEOREM 3. *There is a function $f \in \mathbf{C}^1[0, 1]$ which changes its sign once in $[0, 1]$, such that for any sequence of splines $s_n \in \mathscr{S}(m+1, n)$, which are copositive with f, we have*

$$\limsup_{n \to \infty} \frac{\|f - s_n\|}{\omega_4(f, 1/n)} = \infty. \tag{1.4}$$

Proof. If (1.4) fails, then there is a sequence of splines $s_n \in \mathscr{S}(m+1, n)$, $n = 1, 2, \ldots$, copositive with f and such that

$$\|f - s_n\| \leqslant C\omega_4(f, 1/n), \tag{1.5}$$

hence

$$\omega_4(s_n, 1/n) \leqslant C\omega_4(f, 1/n). \tag{1.6}$$

Now (1.3) together with (1.6) and Theorem 2 contradicts Theorem B. Note that the degree of approximation of f by splines of order $\leqslant 3$ is, in general not better than ω_3 so that (1.5) necessarily implies that s_n is at least cubic. ∎

In fact, Theorems 2 and B show that this type of controlled approximation is impossible even for a much larger class of functions, as stated in Theorem 4 below. We will omit the details of the proof.

THEOREM 4. *There is a function* $f \in \mathbf{C}^1[0, 1]$ *which changes its sign once in* $[0, 1]$, *such that*

$$\limsup_{n \to \infty} \frac{\|f - g_n\| + n^{-1}\omega_r(g_n', 1/n)}{\omega_4(f, 1/n)} = +\infty$$

for any $r \geq 1$ *and any sequence of functions* $g_n \in \mathbf{C}^1[0, 1]$ *that are copositive with* f. *In particular taking* $g_n := f$, $n = 1, 2, ...,$ *we see that for this* f *we have*

$$\limsup_{n \to \infty} \frac{\omega_r(f', 1/n)}{n\omega_4(f, 1/n)} = \infty.$$

We should point out that there is an obvious gap between the affirmative and negative results, which we are not able to bridge at this stage.

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

2. COPOSITIVE SPLINES

In this section we will express the splines as series of B-splines and in particular we will use the Schoenberg–Bernstein variation diminishing operator. For any $n > 0$, let $h := 1/n$, $x_{-2} := x_{-1} := 0$, $x_i := ih$, $i = 0, ..., n$, $x_{n+1} := x_{n+2} := 1$, and

$$\mathbf{T} := \{x_i\}_{i=-2}^{n+2}. \tag{2.1}$$

Then $\{N_{i,3}\}_{i=-2}^{n-1}$ forms a basis for all quadratic splines on $[0, 1]$ with knot sequence $\mathbf{T}$, where

$$N_{i,m}(x) := (x_{i+m} - x_i)[x_i, ..., x_{i+m}](\cdot - x)_+^{m-1}$$

are the B-splines of order m on $\mathbf{T}$. Here, $[x_i, ..., x_{i+m}]\, g(\cdot)$ denotes the mth order divided difference of g. We define $J_i := [x_i, x_{i+1}]$ and set

$$\bar{x}_i := \frac{x_i + x_{i+1}}{2} = \begin{cases} x_0, & i = -1 \\ x_i + 0.5h, & i = 0, ..., n-1 \\ x_n, & i = n \end{cases} \tag{2.2}$$

to be the knot averages. Since the interior knots are equally spaced, we also have

$$x_{i+1} = \frac{\bar{x}_i + \bar{x}_{i+1}}{2}, \qquad i = 0, ..., n-2. \tag{2.3}$$

In addition to well known facts about general B-splines, we will need the following specific properties about the B-splines $N_{i,3}$ (see, e.g., [sch], [dev-lor]).

> For $i = 0, ..., n-3$, $N_{i,3}$ is symmetric about the line $x = \bar{x}_{i+1}$, increasing on the left and decreasing on the right. It assumes $\frac{1}{8}$ at $\bar{x}_i$ and $\bar{x}_{i+2}$, $\frac{1}{2}$ at x_{i+1} and x_{i+2}, and the maximum value $\frac{3}{4}$ at the center $\bar{x}_{i+1}$ of its support $[x_i, x_{i+3}]$. It is convex on J_i and J_{i+2} and concave on J_{i+1}. (2.4.1)

$$N_{-1,3}(x_1) = N_{n-2,3}(x_{n-1}) = \tfrac{1}{2};\; N_{-1,3}(\bar{x}_1) = N_{n-2,3}(\bar{x}_{n-2}) = \tfrac{1}{8}. \tag{2.4.2}$$

$$\sum_{i=-2}^{n-1} c_i N_{i,3}(x_j) = \begin{cases} c_{-2}, & j = 0 \\ (c_{j-1} + c_{j-2})/2, & j = 1, ..., n-1 \\ c_{n-1}, & j = n. \end{cases} \tag{2.4.3}$$

The Schoenberg–Bernstein operator $S_{\mathbf{T}}$ is given by

$$S_{\mathbf{T}}(f, x) := \sum_{i=-2}^{n-1} f(\bar{x}_{i+1})\, N_{i,3}(x). \tag{2.5}$$

Denote $\tilde{s} := S_{\mathbf{T}}(f)$; then by (2.4.3) we have

$$\tilde{s}(x_i) = \begin{cases} f(x_0), & i = 0 \\ (f(\bar{x}_i) + f(\bar{x}_{i-1}))/2, & i = 1, ..., n-1 \\ f(x_n), & i = n. \end{cases} \tag{2.6}$$

Recalling the differentiation formula for B-splines,

$$\frac{d}{dx}\left(\sum_i c_i N_{i,m}(x)\right) = (m-1) \sum_i \frac{c_i - c_{i-1}}{x_{i+m-1} - x_i} N_{i,m-1}(x), \tag{2.7}$$

we obtain

$$\tilde{s}' = \sum_{i=-1}^{n-1} c_{i,1} N_{i,2}, \tag{2.8}$$

with

$$c_{i,1} := \frac{f(\bar{x}_{i+1}) - f(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i}, \qquad i = -1, ..., n-1. \tag{2.9}$$

Note that

$$\tilde{s}'(x_i) = c_{i-1,1}, \qquad i = 0, ..., n, \tag{2.10}$$

since $N_{j-1,2}(x_i)$ equals 1 for $j = i$ and equals 0 for other j. The points $P_i(\bar{x}_i, f(\bar{x}_i))$, $i = -1, ..., n$, on the graph of f totally determine the graph of $\tilde{s}$ and are called the control points of $\tilde{s}$ [dev-lor]. In terms of the control points, (2.6) and (2.10) can be geometrically restated as: for each $i = 0, ..., n$, the line segment $\overline{P_{i-1}P_i}$ is tangent to the graph of $\tilde{s}$ at $x = x_i$.

We now give the proof of Theorem 1, throughout which we denote by C an absolute constant which may be different on different occurrences, even in the same line, and for the sake of brevity, we denote $\omega_2(f, 1/n)$ by ω_2.

Proof of Theorem 1. We fix $n \geqslant 4/\delta$ and we apply the Schoenberg–Bernstein operator $S_{\mathbf{T}}$ to f to obtain the quadratic spline $\tilde{s}$ on $\mathbf{T}$ given in (2.5). It is known [dev-lor] that

$$\|f - \tilde{s}\| \leqslant C\omega_2. \tag{2.11}$$

Let $I_i := (\bar{x}_i, \bar{x}_{i+1}]$, $i = -1, ..., n-1$. We call I_p contaminated if $\bar{x}_p < y_j \leqslant \bar{x}_{p+1}$ for some j. Since $n \geqslant 4/\delta$, there is exactly one y_j in each contaminated interval. Moreover, if I_p and I_l are any two consecutive contaminated intervals, then

$$-1 \leqslant p < p+4 \leqslant l \leqslant n-1; \tag{2.12}$$

that is, the distance between them is at least $3h$. For any $x \in [x_{p+2}, x_l]$,

$$\tilde{s}(x) = \sum_{i=-2}^{n-1} f(\bar{x}_{i+1})\, N_{i,3}(x) = \sum_{i=p}^{l-1} f(\bar{x}_{i+1})\, N_{i,3}(x) \tag{2.13}$$

has the same sign as f, since f does not change sign on $[\bar{x}_{p+1}, \bar{x}_l]$. Similarly, $\tilde{s}$ has the same sign as f on $[x_0, x_p]$ if I_p is the first contaminated interval and $p \neq -1$, and on $[x_{p+2}, x_n]$ if I_p is the last such interval and $p \neq n-1$. This means $\tilde{s}$ may have the wrong sign only near a point of sign change.

We are ready to modify $\tilde{s}$ so that the resulting spline s should be copositive with f, on the whole interval $[0, 1]$. Let y_j be any of the points of sign change, and I_p be the contaminated interval containing y_j. We may have to correct the sign of $\tilde{s}$ near y_j; thus we add to it a correction function

$$m_j(x) := \varepsilon\alpha N(x), \tag{2.14}$$

where $\varepsilon := -\tilde{s}(y_j)$, N is a translation of $N_{0,3}$ (restricted to $[0, 1]$ if necessary) to be prescribed later, and $\alpha := 1/N(y_j) > 0$. The support of any correction function contains the corresponding contaminated interval in its interior and extends beyond it by no more than $1.5h$ on each side. Therefore, by (2.12), the interiors of the supports of the correction functions m_j, $j = 1, ..., k$, are mutually disjoint. This anables us to define s by

$$s(x) = \begin{cases} \tilde{s}(x) + m_j(x), & \text{if } x \text{ is in the support of some } m_j \\ \tilde{s}(x), & \text{otherwise.} \end{cases} \tag{2.15}$$

For each point of sign change y_j, we have

$$s(y_j) = \tilde{s}(y_j) + m_j(y_j) = \tilde{s}(y_j) - \frac{\tilde{s}(y_j)}{N(y_j)} N(y_j) = 0 = f(y_j), \tag{2.16}$$

and it is obvious that in order to conclude that s is compositive with f, we only have to show that they are copositive in the supports of the various m_j's. We will assume that in I_p, $f \geqslant 0$ to the left of y_j, and that $f \leqslant 0$ to the right. The proof in the other case follows by replacing f by $-f$. The modification is done in the following five cases.

Case 1. $p = -1$ and $\tilde{s}(y_j) \leqslant 0$. Recall $I_{-1} = (\bar{x}_{-1}, \bar{x}_0] = (x_0, \bar{x}_0]$. We take $N(x) := N_{0,3}(x + 2h)$ restricted to $[0, 1]$. Note $\varepsilon = -\tilde{s}(y_j) \geqslant 0$. By (2.4.1), m_j is decreasing on its support $J_0 = [x_0, x_1]$, and $N(x_0) = \frac{1}{2} > N(y_j) \geqslant N(\bar{x}_0) = \frac{1}{8}$, thus $2 < \alpha \leqslant 8$. By (2.6) we have

$$s(x_0) = \tilde{s}(x_0) + m_j(x_0) \geqslant f(x_0) \geqslant 0,$$

$$s(x_1) = \tilde{s}(x_1) + m_j(x_1) = \frac{f(\bar{x}_0) + f(\bar{x}_1)}{2} \leqslant 0.$$

Since s is a parabola on J_0, these together with (2.16) guarantee that s is copositive with f on that interval.

Case 2. $p = -1$ and $\tilde{s}(y_j) > 0$. In this case $\varepsilon = -\tilde{s}(y_j) < 0$. We take $N(x) := N_{0,3}(x + h)$ restricted to $[0, 1]$. The monotonicity of N on $[x_0, \bar{x}_0]$ gives $\frac{4}{3} \leqslant \alpha < 2$. We need to show that s and f are copositive on $[x_0, x_2]$, the support on m_j. It is trivial to do so on $J_1 = [x_1, x_2]$, since by (2.13), $\tilde{s} \leqslant 0$ and so are m_j and f. In particular, $s(x_1) \leqslant 0$.

Since by assumption, $f(x_0) \geqslant 0$ and $f(\bar{x}_0) \leqslant 0$, we have

$$\tilde{s}'(x_0) = \frac{f(\bar{x}_0) - f(x_0)}{h/2} \leqslant 0.$$

By (2.4.1) we also have $m_j'(x_0) \leqslant 0$, thus $s'(x_0) \leqslant 0$. since s is a parabola on J_0 with $s'(x_0) \leqslant 0$, $s(y_j) = 0$, and $s(x_1) \leqslant 0$, one can readily see that s is also copositive with f on J_0.

Case 3. $p=n-1$. The proof follows from Cases 1 and 2 by replacing $f(x)$ by $-f(1-x)$.

Case 4. $0 \leqslant p \leqslant n-2$, $\tilde{s}(y_j) \leqslant 0$. We will again take full advantage of the fact that s is a parabola on each interval J_i. This time we use $N := N_{p-1,3}$. Note that $\varepsilon = -\tilde{s}(y_j) \geqslant 0$ and $\frac{4}{3} < \alpha \leqslant 8$. We need to show that s and f are copositive on $[x_{p-1}, x_{p+2}]$. This is obvious on J_{p-1} since $\tilde{s}$, m_j, and f are all nonnegative. (This is still true even if $p=0$, although J_{-1} only consists of a single point.)

For the interval J_{p+1}, we claim that both $\tilde{s}$ and $s = \tilde{s} + m_j$ are either convex or decreasing there. We only need to show this for $\tilde{s}$ because m_j is both convex and decreasing in J_{p+1}. Indeed, since $\tilde{s}'$ is linear on this interval, by (2.10), $\tilde{s}$ is coconvex with the broken line $\overline{P_p P_{p+1} P_{p+2}}$, where $P_i(\bar{x}_i, f(\bar{x}_i))$, $i = p, p+1$ and $p+2$, are control points defined earlier. If $\overline{P_p P_{p+1} P_{p+2}}$ is concave, then the slope of $\overline{P_{p+1} P_{p+2}}$ is smaller than that of $\overline{P_p P_{p+1}}$, which is already nonpositive (by our assumption on how f changes its sign in I_p). This means $\tilde{s}'$ is nonpositive at both endpoints x_{p+1} and x_{p+2}, hence on the whole interval, and our claim follows.

On J_p we claim that s is either concave or decreasing. An argument similar to the above shows this is true on the interval $[\bar{x}_p, x_{p+1}]$. If s is convex on I_p, where it is a parabola, then it must be decreasing on $[\bar{x}_p, x_{p+1}] \subset J_p$, and therefore it is decreasing on whole J_p.

Now if $x_{p+1} \leqslant y_j \leqslant \bar{x}_{p+1}$, then these two claims together with $s(x_p) \geqslant 0$, $s(y_j) = 0$, and $s(x_{p+2}) \leqslant 0$ (see (2.13)) imply that it is copositive with f on J_{p+1} and that it is nonnegative on J_p. If $\bar{x}_p < y_j < x_{p+1}$, then the above imply that s is copositive with f on J_p and that s is non-positive on J_{p+1}.

Case 5. $0 \leqslant p \leqslant n-2$ and $\tilde{s}(y_j) > 0$. The proof follows from Case 4 by replacing $f(x)$ by $-f(1-x)$.

Now that we have shown that the modified spline s is copositive with f on [0, 1], it remains to prove that it satisfies (1.1). By (2.15) it suffices to show that

$$\|m_j\| \leqslant Cn\omega_2, \tag{2.17}$$

since then, (1.1) will follow from (2.11). But this is obvious since for all cases in the modification we have $|\alpha| \leqslant C$ and

$$|\varepsilon| = |-\tilde{s}(y_j)| = |f(y_j) - \tilde{s}(y_j)| \leqslant C\omega_2. \quad \blacksquare$$

3. Copositive Polynomials

In this section we will prove Theorem 2. We first establish two lemmas.

Lemma 1. *There is a polynomial $q_n(x)$ of degree $\leqslant 2n$ which is odd and increasing in $[-1, 1]$ and such that*

$$|q_n(x)| \leqslant 1, \qquad \textit{for} \quad |x| \leqslant 1, \tag{3.1}$$

$$q_n'(x) \geqslant An, \qquad \textit{for} \quad |x| \leqslant \frac{1}{n}, \tag{3.2}$$

and

$$q_n'(x) \leqslant \frac{B}{nx^2}, \qquad \textit{for} \quad \frac{1}{n} < |x| \leqslant 1, \tag{3.3}$$

where A and B are absolute constants.

Proof. Define

$$q_n(x) := c_n \int_{-1}^{x} \frac{\sin^2((n/2)\arccos(1-t^2/2))}{\sin^2 \frac{1}{2}\arccos(1-t^2/2)}\, dt - \frac{1}{2}, \qquad -1 \leqslant x \leqslant 1,$$

where

$$c_n^{-1} := \int_{-1}^{1} \frac{\sin^2((n/2)\arccos(1-t^2/2))}{\sin^2 \frac{1}{2}\arccos(1-t^2/2)}\, dt.$$

It is readily seen that $c_n \sim n^{-1}$ and that q_n is a polynomial of degree $\leqslant 2n$ which is odd and increasing in $[-1, 1]$ and satisfies (3.1). Also,

$$q_n'(x) = c_n \frac{\sin^2((n/2)\arccos(1-x^2/2))}{\sin^2 \frac{1}{2}\arccos(1-x^2/2)}, \qquad -1 \leqslant x \leqslant 1.$$

If $|x| \leqslant 1/n$, then $(n/2)\arccos(1-x^2/2) \leqslant \pi/2$, so that applying $2\theta/\pi \leqslant \sin\theta$, $0 \leqslant \theta \leqslant \pi/2$, it follows that

$$\frac{\sin^2((n/2)\arccos(1-x^2/2))}{\sin^2 \frac{1}{2}\arccos(1-x^2/2)} \geqslant \left(\frac{(2/\pi)(n/2)\arccos(1-x^2/2)}{\frac{1}{2}\arccos(1-x^2/2)}\right)^2 = \frac{4}{\pi^2} n^2.$$

Thus

$$q_n'(x) \geqslant An.$$

If $1/n \leqslant |x| \leqslant 1$, we have

$$\frac{\sin^2((n/2)\arccos(1-x^2/2))}{\sin^2 \frac{1}{2}\arccos(1-x^2/2)} \leqslant \frac{1}{((1/\pi)\arccos(1-x^2/2))^2} \leqslant \frac{B'}{x^2}$$

and so

$$q_n'(x) \leqslant \frac{B}{nx^2}. \quad \blacksquare$$

LEMMA 2. *Let $f \in \mathbf{C}^1[0, 1]$ and let $y_0 := 0 < y_1 < y_2 < \cdots < y_k < 1 =: y_{k+1}$. Denote $\delta := \min_{0 \leqslant i \leqslant k}(y_{i+1} - y_i)$, and let $r \geqslant 1$. Then for each $n \geqslant k$ there exists a polynomial p_n of degree $\leqslant n$ such that*

$$p_n(y_i) = f(y_i), \qquad i = 1, 2, ..., k, \tag{3.4}$$

$$\|f - p_n\| \leqslant Cn^{-1}\omega_r(f', 1/n), \tag{3.5}$$

and

$$\|f' - p_n'\| \leqslant C\omega_r(f', 1/n), \tag{3.6}$$

where C depends only on k, r, and δ.

Proof. Since $f \in \mathbf{C}^1[0, 1]$, it is well known that there is a polynomial $\tilde{p}_n$ of degree $\leqslant n$, which simultaneously satisfies

$$\|f - \tilde{p}_n\| \leqslant Cn^{-1}\omega_r(f', 1/n),$$

and

$$\|f' - \tilde{p}_n'\| \leqslant C\omega_r(f', 1/n).$$

(We are applying here a very weak version of the estimates for simultaneous approximation by polynomials. It seems that such estimates involving ω_r with $r \geqslant 2$ first appeared in a paper by Gopengauz [gop], who claims that they follow from a result of Brudnyi using some ideas of Trigub.) Now define $p_n(x) := \tilde{p}_n(x) + \tilde{p}_k(x)$, where $\tilde{p}_k$ is the polynomial of degree $k-1$ interpolating $f(x) - \tilde{p}_n(x)$ at y_i $(i = 1, 2, ..., k)$. Then p_n has the deisred properties. $\blacksquare$

We are ready to prove Theorem 2.

Proof of Theorem 2. Given any fixed $n > C_1\delta^{-1}$, where C_1 is a constant still to be prescribed, we shall show that there exists a polynomial P_n copositive with f of degree $\leqslant 2kn$, satisfying (1.2). It is then a standard thing to obtain a polynomial of degree n satisfying (1.2). Let p_n be the polynomial from Lemma 2 and set

$$P_n(x) := p_n(x) + \varepsilon D C n^{-1} \omega_r(f', 1/n) \prod_{i=1}^{k} q_n(x - y_i), \tag{3.7}$$

where q_n is given in Lemma 1, $\varepsilon = \operatorname{sgn} f(x)$ for $x \in (y_k, 1)$, C is the constant in Lemma 2, and D is a positive constant to be determined later.

Since q_n is odd, increasing and $q_n'(x) \geq An$ for $|x| \leq 1/n$, we have

$$|q_n(x)| \geq A, \qquad |x| > 1/n.$$

Hence

$$\left| \prod_{i=1}^{k} q_n(x - y_i) \right| \geq A^k, \qquad x \notin \bigcup_{i=1}^{k} \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right]. \tag{3.8}$$

Note that the second term on the right-hand side of (3.7) is copositive with f. If we take $D > A^{-k}$, then by (3.5) and ((3.8), we obtain

$$f(x)\, P_n(x) \geq 0, \qquad x \notin \bigcup_{i=1}^{k} \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right]. \tag{3.9}$$

On the other hand, it follows from (3.4) and (3.7) that

$$f(y_i) - P_n(y_i) = 0, \qquad i = 1, 2, ..., k. \tag{3.10}$$

We will show that if f changes from $-$ to $+$ at y_i, then

$$f'(x) - P_n'(x) \leq 0, \qquad x \in \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right], \tag{3.11}$$

and if f changes from $+$ to $-$ at y_i, then

$$f'(x) - P_n'(x) \geq 0, \qquad x \in \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right]. \tag{3.12}$$

Then, if $f(x) \leq 0$ for $x \in (y_i - 1/n, y_i)$ and $f(x) \geq 0$ for $x \in (y_i, y_i + 1/n)$, then by (3.10) and (3.11), there exists a number ξ between x and y_i such that

$$\begin{aligned} f(x) - P_n(x) &= [f(x) - P_n(x)] - [f(y_i) - P_n(y_i)] \\ &= (x - y_i)[f'(\xi) - P_n'(\xi)], \end{aligned} \tag{3.13}$$

which gives

$$f(x)-P_n(x)\geqslant 0, \qquad x\in\left(y_i-\frac{1}{n}, y_i\right)$$

and

$$f(x)-P_n(x)\leqslant 0, \qquad x\in\left(y_i, y_i+\frac{1}{n}\right).$$

On the other hand if $f(x)\geqslant 0$ for $x\in(y_i-1/n, y_i)$ and $f(x)\leqslant 0$ for $x\in(y_i, y_i+1/n)$, then by (3.12) and (3.13), we have

$$f(x)-P_n(x)\leqslant 0, \qquad x\in\left(y_i-\frac{1}{n}, y_i\right)$$

and

$$f(x)-P_n(x)\geqslant 0, \qquad x\in\left(y_i, y_i+\frac{1}{n}\right).$$

Therefore, for $x\in\bigcup_{i=1}^k [y_i-1/n, y_i+1/n]$, we either have $P_n(x)\geqslant f(x)\geqslant 0$ or $P_n(x)\leqslant f(x)\leqslant 0$, i.e.,

$$f(x)\,P_n(x)\geqslant 0, \qquad x\in\bigcup_{i=1}^k\left[y_i-\frac{1}{n}, y_i+\frac{1}{n}\right]. \tag{3.14}$$

To this end, we will show that for $D>2A^{-k}$, (3.11) and (3.12) hold. We have

$$P_n'(x)=p_n'(x)+\varepsilon DCn^{-1}\omega_r(f', 1/n)\left(\prod_{j=1}^k q_n(x-y_j)\right)' \tag{3.15}$$

and

$$\begin{aligned}\left(\prod_{j=1}^k q_n(x-y_j)\right)' &= q_n'(x-y_i)\prod_{\substack{j=1\\ j\neq i}}^k q_n(x-y_j)\\ &\quad+q_n(x-y_i)\left[\prod_{\substack{j=1\\ j\neq i}}^k q_n(x-y_j)\right]'\\ &=: I_1(x)+I_2(x).\end{aligned} \tag{3.16}$$

By (3.2) and the analogue of (3.8) for $k-1$, it follows that

$$|I_1(x)|\geqslant AnA^{k-1}=A^kn, \qquad x\in\left[y_i-\frac{1}{n}, y_i+\frac{1}{n}\right]. \tag{3.17}$$

Also by (3.1) and (3.3), for $x \in [y_i - 1/n, y_i + 1/n]$ and $n > 3/\delta$ we have

$$|I_2(x)| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{B}{n(x-y_j)^2} \leqslant \frac{B(k-1)}{n(2\delta/3)^2} \leqslant \frac{9Bk}{4n\,\delta^2}. \tag{3.18}$$

Now let $C_1 := \max(3, 3\sqrt{Bk/2A^k})$. Since $n > C_1 \delta^{-1}$, then by (3.17) and (3.18),

$$|I_2(x)| \leqslant \frac{9Bk}{4nC_1^2 n^{-2}} \leqslant \frac{1}{2} A^k n,$$

and

$$|I_1(x)| - |I_2(x)| \geqslant \tfrac{1}{2} A^k n. \tag{3.19}$$

Hence, if $n > C_1 \delta^{-1}$ and $D > 2A^{-k}$, the second term in the right-hand side of (3.15) has an absolute value greater than $C\omega_r(f', 1/n)$. Thus by (3.6), the sign of $f'(x) - P'_n(x)$ is determined by the sign of

$$-\varepsilon q'_n(x-y_i) \prod_{\substack{j=1 \\ j \neq i}}^{k} q_n(x-y_j).$$

Since

$$q'_n(x-y_i) \geqslant 0$$

and

$$\operatorname{sgn}\left[\varepsilon \prod_{j=1}^{k} q_n(x-y_j)\right] = \operatorname{sgn} f(x)$$

for $x \in [0, 1]$, we have

$$\operatorname{sgn}\left[-\varepsilon q'_n(x-y_i) \prod_{\substack{j=1 \\ j \neq i}}^{k} q_n(x-y_j)\right] = \operatorname{sgn}[-q_n(x-y_i)\, f(x)],$$

and (3.11) and (3.12) follow. Combining (3.9) and (3.14) we see that P_n and f are copositive in [0, 1]. It is clear from our proof that the constant C_2 such that (1.2) holds depends only on k, r, and δ. This completes the proof. ∎

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